

CLASSIFICATION OF ISOLATED SINGULARITIES OF NONNEGATIVE SOLUTIONS TO FRACTIONAL SEMI-LINEAR ELLIPTIC EQUATIONS AND THE EXISTENCE RESULTS

HUYUAN CHEN ALEXANDER QUAAS

Abstract. In this paper, we classify the singularities of nonnegative solutions to fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1}$$

where $p > 1$, Ω is a bounded, C^2 domain in \mathbb{R}^N containing the origin, $N \geq 2$ and the fractional Laplacian $(-\Delta)^\alpha$ is defined in the principle value sense. We obtain that any classical solution u of (1) is a weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{aligned} \tag{2}$$

for some $k \geq 0$, where δ_0 is the Dirac mass at the origin. In particular, when $p \geq \frac{N}{N-2\alpha}$, we have that $k = 0$; when $p < \frac{N}{N-2\alpha}$, u has removable singularity at the origin if $k = 0$ and if $k > 0$, u satisfies

$$\lim_{x \rightarrow 0} u(x)|x|^{N-2\alpha} = c_{N,\alpha}k,$$

where $c_{N,\alpha} > 0$.

Furthermore, when $p \in (1, \frac{N}{N-2\alpha})$, we obtain that there exists $k^* > 0$ such that problem (1) has at least two positive solutions for $k < k^*$, a unique positive solution for $k = k^*$ and no positive solution for $k > k^*$.

1. INTRODUCTION

Our purpose of this paper is to classify the singularities of nonnegative solutions of fractional semi-linear elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.1}$$

where $p > 1$, Ω is a C^2 bounded domain in \mathbb{R}^N containing the origin, $N \geq 2$, the fractional Laplacian $(-\Delta)^\alpha$ is defined in the principle value sense, i.e.

$$(-\Delta)^\alpha u(x) = c_{N,\alpha} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(z)}{|x - z|^{N+2\alpha}} dz.$$

Here $B_\epsilon(0)$ is the ball with radius ϵ centered at the origin and $c_{N,\alpha} > 0$ is the normalized constant, see [25].

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When $\alpha = 1$, $-\Delta$ is the well-known Laplace operator and the related isolated singular problem

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega \setminus \{0\} \\ u &\geq 0 \quad \text{in } \Omega \setminus \{0\}, \quad u = 0 \quad \text{on } \partial\Omega \end{aligned} \quad (1.2)$$

has been classified by Lions in [26] for $p \in (1, \frac{N}{N-2})$, by Aviles in [1] for $p = \frac{N}{N-2}$, by Gidas and Spruck in [19] for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, by Caffarelli, Gidas and Spruck in [9] for $p = \frac{N+2}{N-2}$. When $p \in (1, \frac{N}{N-2})$, Lions in [26] showed that any nonnegative solution of (1.2) is a very weak solution of

$$\begin{aligned} -\Delta u &= u^p + k\delta_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (1.3)$$

for some $k \geq 0$, and further noted that there exists $k^* > 0$ such that for $k \in (0, k^*)$, problem (1.3) has at least two solutions including the minimal solution and a Mountain Pass type solution; for $k = k^*$, problem (1.3) has a unique solution; there is no solution of (1.3) for $k > k^*$. So the solution of (1.2) has either the singularity of $|x|^{2-N}$ or removable singularity when $p \in (1, \frac{N}{N-2})$. In contrast with problem (1.3) with source nonlinearity, Véron in [36] showed that the semi-linear elliptic equations with absorption terms

$$\begin{aligned} -\Delta u + u^p &= 0 \quad \text{in } \Omega \setminus \{0\}, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.4)$$

admits positive solutions, when $p \in (1, \frac{N}{N-2})$, which satisfy

$$\text{either } \lim_{x \rightarrow 0} u(x)|x|^{N-2} = c_N k \quad \text{or} \quad \lim_{x \rightarrow 0} u(x)|x|^{\frac{2}{p-1}} = c_p > 0$$

for $k > 0$, denoting by u_k and u_∞ respectively. Furthermore, u_∞ is the limit of $\{u_k\}_k$ as $k \rightarrow +\infty$ and u_k is a weak solution of

$$\begin{aligned} -\Delta u + u^p &= k\delta_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.5)$$

Such an object has been extended to the equations with Radon measures or boundary measure data in [5, 8, 20, 22, 23] and more related topics see references [2, 3, 7, 37].

When $\alpha \in (0, 1)$, $(-\Delta)^\alpha$ is a non-local operator, which has been studied by Caffarelli and Sivestre in [11, 12, 14], and fractional equations with measures and absorption nonlinearity in type (1.5) have been studied by Chen and Véron in [16, 17]. In the source nonlinearity case, Chen, Felmer and Véron in [15] obtained one solution for

$$\begin{aligned} (-\Delta)^\alpha u &= g(u) + \sigma\nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $\sigma > 0$ small, ν is a Radon measure and nonnegative function g satisfies the integral subcritical condition

$$\int_1^{+\infty} g(s)s^{-1-\frac{N}{N-2\alpha}} ds < +\infty.$$

Our interest in this paper is to classify the singularities of (1.1) and then to obtain the existence of singular solutions of (1.1) by considering the very weak solutions of corresponding problem with Dirac mass.

The classification of singularities of nonnegative solutions for (1.1) states as follows.

Theorem 1.1. *Assume that $p > 1$ and u is a nonnegative classical solution of (1.1). Then $u \in L^p(\Omega)$ and there exists $k \geq 0$ such that u is a very weak solution of*

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \quad (1.6)$$

that is, $u \in L^p$ and

$$\int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\Omega), \quad (1.7)$$

where $C_c^\infty(\Omega)$ is the space of all the functions in $C^\infty(\mathbb{R}^N)$ with the support in Ω . Furthermore,

- (i) When $p \geq \frac{N}{N-2\alpha}$, we have that $k = 0$.
- (ii) When $p \in (1, \frac{N}{N-2\alpha})$, if $k = 0$, u is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega; \end{aligned} \quad (1.8)$$

if $k > 0$, then u satisfies

$$\lim_{x \rightarrow 0} u(x)|x|^{N-2\alpha} = c_{N,\alpha}k. \quad (1.9)$$

Notice that for $\alpha = 1$, by the local property of the Laplacian and Integration by Part formula, the solution u of (1.2) has the following essential estimate of the singularity of the average in sphere of the corresponding solution

$$\bar{u}(r) \leq \frac{c_1}{r^{N-2}}, \quad r > 0 \text{ small},$$

where $c_1 > 0$ and $\bar{u}(r) = \int_{\partial B_r(0)} u(x) d\omega(x)$, then it is available to apply the Schwartz's Theorem in [30] to classify the singularity of solutions of (1.2). However, for $\alpha \in (0, 1)$, because of the nonlocal property of the fractional Laplacian, problem (1.1) can not be translated into ODE by the average sphere function. The strategy to prove Theorem 1.1 is to derive $u \in L^p(\Omega)$ and to scale the typical test functions and by using the positiveness of the solution to derive that

$$L(\xi) := \int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\Omega).$$

We notice that $k = 0$ in the super critical case, i.e. $p \geq \frac{N}{N-2\alpha}$, which means that the singularity of positive solution is not visible in the distribution sense.

From Theorem 1.1, the solution of (1.1) may have the singularity as $|x|^{2\alpha-N}$ or removable singularity at the origin. Next we consider the existence and nonexistence singular solution of (1.1) by dealing with the very weak solutions to (1.6) when $p \in (1, \frac{N}{N-2\alpha})$.

Theorem 1.2. *Assume that $p \in (1, \frac{N}{N-2\alpha})$, then there exists $k^* > 0$, such that*

- (i) for $k \in (0, k^*)$, problem (1.6) admits a minimal positive solution u_k and a Mountain-Pass type solution $w_k > u_k$, both solutions are classical solutions of (1.1) and satisfy (1.9);
- (ii) for $k = k^*$, problem (1.6) admits a unique positive solution u_k , which is a classical solution of (1.1) and satisfies (1.9);
- (iii) for $k \geq k^*$, problem (1.6) admits no solution.

We remark that the minimal positive solution of (1.1) is derived by iterating an increasing sequence $\{v_n\}_n$ defined by

$$v_0 = k\mathbb{G}_\alpha[\delta_0], \quad v_n = \mathbb{G}_\alpha[v_{n-1}^p] + k\mathbb{G}_\alpha[\delta_0],$$

where $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined as

$$\mathbb{G}_\alpha[f](x) = \int_{\Omega} G_\alpha(x, y) f(y) dy$$

and G_α is the Green kernel of $(-\Delta)^\alpha$ in $\Omega \times \Omega$. The properties of Green's function see Theorem 1.1 in [18]. To insure the convergence of the sequence $\{v_n\}_n$, we need to construct a suitable barrier function by using the estimate

$$\mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] \leq c_2 \mathbb{G}_\alpha[\delta_0] \quad \text{in } \Omega \setminus \{0\},$$

where $c_2 > 0$. By the analysis the stability of the minimal solution, we deduce the existence of the very weak solution in the case that $k = k^*$ and for $k \in (0, k^*)$, we construct Mountain Pass solution v_k for the problem

$$\begin{aligned} (-\Delta)^\alpha u &= (u_k + u_+)^p - u_k^p & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{aligned}$$

and then the Mountain Pass type solution $v_k + u_k$ is a solution of (1.6).

The paper is organized as follows. In Section 2, we show the integrability of the solution u of (1.1) and the isolated support of operator generated by $(-\Delta)^\alpha u - u^p$. Section 3 is devoted to do classification of the singularities of (1.1). Finally, in Section 4 we prove the existence and nonexistence of very weak solutions of problem (1.6).

2. PRELIMINARY RESULTS

We start our analysis from the integrality of nonnegative solution u to the fractional problem (1.1). In what follows, denote by c_i the positive constant with $i \in \mathbb{N}$, G_α denotes the Green's function of $(-\Delta)^\alpha$ in $\Omega \times \Omega$ and $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined as

$$\mathbb{G}_\alpha[f](x) = \int_{\Omega} G_\alpha(x, y) f(y) dy.$$

Proposition 2.1. *Assume that $p > 1$ and u is a nonnegative classical solution of (1.1). Then*

$$u \in L^p(\Omega). \tag{2.1}$$

Proof. If $u^p \notin L^1(\Omega)$, then it implies by $u \in L_{loc}^\infty(\Omega \setminus \{0\})$ that

$$\lim_{r \rightarrow 0^+} \int_{\Omega \setminus B_r(0)} u^p dx = +\infty.$$

So for any $r > 0$, there exist decreasing sequence $\{R_n\}_n$ such that $R_n \in (0, r)$, $\lim_{n \rightarrow \infty} R_n = 0$ and

$$\int_{B_r(0) \setminus B_{R_n}(0)} u^p dx = n. \tag{2.2}$$

Let v_n be the solution of

$$\begin{aligned} (-\Delta)^\alpha v_n &= \chi_{\Omega \setminus B_{R_n}(0)} u^p & \text{in } \Omega, \\ v_n &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $\chi_O = 1$ in O and $\chi_O = 0$ in $\mathbb{R}^N \setminus O$ for any domain O in \mathbb{R}^N .

Let Γ_0 be the Fundamental solution of

$$(-\Delta)^\alpha \Gamma_0 = \delta_0 \quad \text{in } \mathbb{R}^N.$$

In fact, $\Gamma_0(x) = c_{N,\alpha}|x|^{2\alpha-N}$ for $x \in \mathbb{R}^N \setminus \{0\}$. Since $u \geq 0$ in $\Omega \setminus \{0\}$, $\lim_{x \rightarrow 0}(u + \Gamma_0)(x) = +\infty$ and v_n is bounded in Ω , then there exists $r > 0$ such that $u + \Gamma_0 \geq v_n$ in $B_r(0) \setminus \{0\}$ and it implies by Comparison Principle [13, Theorem 2.3] that for any $n \in \mathbb{N}$

$$u + \Gamma_0 \geq v_n \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (2.3)$$

Since $\lim_{y \rightarrow x} G(x, y) = +\infty$ for $x \in \Omega$, there exists $r_0 > 0$ such that $G(x, y) \geq 1$ for $x, y \in B_{r_0}(0)$, and by (2.2),

$$\begin{aligned} v_n(x) &= \mathbb{G}_\alpha[\chi_{\Omega \setminus B_{R_n}(0)} u^p] = \int_{\Omega \setminus B_{R_n}(0)} G(x, y) u^p(y) dy \\ &\geq \int_{B_{r_0}(0) \setminus B_{R_n}(0)} u^p(y) dy \\ &= n \rightarrow +\infty \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which together with (2.3) implies that $u + \Gamma_0 = +\infty$ in $B_{r_0}(0)$ and this is impossible. Therefore, we have that $u^p \in L^1(\Omega)$. \square

To improve the regularity, we need following regularity result.

Proposition 2.2. [34, Proposition 1.4]

Let $h \in L^s(\Omega)$ with $s \geq 1$, then, there exists $c_3 > 0$ such that

(i)

$$\|\mathbb{G}_\alpha[h]\|_{L^\infty(\Omega)} \leq c_3 \|h\|_{L^s(\Omega)} \quad \text{if } \frac{1}{s} < \frac{2\alpha}{N}; \quad (2.4)$$

(ii)

$$\|\mathbb{G}_\alpha[h]\|_{L^r(\Omega)} \leq c_3 \|h\|_{L^s(\Omega)} \quad \text{if } \frac{1}{s} \leq \frac{1}{r} + \frac{2\alpha}{N} \quad \text{and } s > 1; \quad (2.5)$$

(iii)

$$\|\mathbb{G}_\alpha[h]\|_{L^r(\Omega)} \leq c_3 \|h\|_{L^1(\Omega)} \quad \text{if } 1 < \frac{1}{r} + \frac{2\alpha}{N}. \quad (2.6)$$

In the searching the second solution of (1.6), Mountain Pass theorem is applied in the Hilbert space $H_0^\alpha(\Omega)$, defined by the closure of $C_c^\infty(\Omega)$ under the norm of

$$\|v\|_\alpha = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{\frac{1}{2}}.$$

The corresponding inner product in $H_0^\alpha(\Omega)$ is given as

$$\langle u, v \rangle_\alpha = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy.$$

Proposition 2.3. For $s \in [0, 2\alpha)$, the embedding:

$$H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega, |x|^{-s} dx)$$

is continuous and compact for

$$q \in [1, \frac{2N - s}{N - 2\alpha}).$$

Proof. From Theorem 6.10 and Theorem 7.1 in [25], it is known that the embedding

$$H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for

$$q \in [1, \frac{2N}{N - 2\alpha}]$$

and is compact for

$$q \in [1, \frac{2N}{N-2\alpha}).$$

By using Hölder inequality, for $q \in [1, \frac{2N-s}{N-2\alpha})$, let $t = \frac{1}{q} \frac{2N}{N-2\alpha}$, then

$$qt = \frac{2N}{N-2\alpha}, \quad \frac{st}{t-1} = \frac{s}{2N-q(N-2\alpha)} < N$$

and

$$\int_{\Omega} \frac{u^q}{|x|^s} dx \leq \left(\int_{\Omega} u^{qt} dx \right)^{\frac{1}{t}} \left(\int_{\Omega} |x|^{-\frac{st}{t-1}} dx \right)^{1-\frac{1}{t}}, \quad (2.7)$$

thus, the embedding $H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega, |x|^{-s} dx)$ is continuous. Now we choose $t_\epsilon = \frac{1}{q} \frac{2N}{N-2\alpha} - \epsilon$ for $\epsilon > 0$ sufficient small, then (2.7) holds with $qt_\epsilon < \frac{2N}{N-2\alpha}$ and $\frac{st_\epsilon}{t_\epsilon-1} < N$, so the embedding $H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega, |x|^{-s} dx)$ is compact for $q \in [1, \frac{2N-s}{N-2\alpha})$. \square

Lemma 2.1. *Let $\tau \in (0, N)$, then for $x \in B_{\frac{1}{2}}(0) \setminus \{0\}$,*

$$\mathbb{G}_\alpha[|\cdot|^{-\tau}](x) \leq \begin{cases} c_2|x|^{-\tau+2\alpha} & \text{if } \tau > 2\alpha, \\ -c_2 \log(|x|) & \text{if } \tau = 2\alpha, \\ c_2 & \text{if } \tau < 2\alpha. \end{cases} \quad (2.8)$$

For $p \in (1, \frac{N}{N-2\alpha})$, there holds

$$\mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] \leq \begin{cases} c_2|x|^{p(2\alpha-N)+2\alpha} & \text{if } p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha}), \\ -c_2 \log(|x|) & \text{if } p = \frac{2\alpha}{N-2\alpha}, \\ c_2 & \text{if } p < \frac{2\alpha}{N-2\alpha} \end{cases} \quad (2.9)$$

and

$$\mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] \leq c_2 \mathbb{G}_\alpha[\delta_0] \quad \text{in } \Omega \setminus \{0\}. \quad (2.10)$$

Proof. For $x \in B_{\frac{1}{2}}(0) \setminus \{0\}$, we have that

$$\begin{aligned} \mathbb{G}_\alpha[|\cdot|^{-\tau}](x) &\leq c_{N,\alpha} \int_{B_{R_0}(0)} \frac{1}{|x-y|^{N-2\alpha}} \frac{1}{|y|^\tau} dy \\ &= c_{N,\alpha} |x|^{2\alpha-\tau} \int_{B_{\frac{R_0}{|x|}}(0)} \frac{1}{|e_x-y|^{N-2\alpha}} \frac{1}{|y|^\tau} dy \\ &\leq c_4 |x|^{2\alpha-\tau} \int_{B_{\frac{R_0}{|x|}}(0)} \frac{1}{1+|y|^{N-2\alpha+\tau}} dy \\ &\leq \begin{cases} c_2|x|^{-\tau+2\alpha} & \text{if } \tau > 2\alpha, \\ -c_2 \log(|x|) & \text{if } \tau = 2\alpha, \\ c_2 & \text{if } \tau < 2\alpha, \end{cases} \end{aligned}$$

where $e_x = \frac{x}{|x|}$ and $R_0 > 0$ such that $\Omega \subset B_{R_0}(0)$.

From [18], we know that

$$\mathbb{G}_\alpha[\delta_0](x) \leq \frac{c_{N,\alpha}}{|x|^{N-2\alpha}}$$

and

$$\mathbb{G}_\alpha^p[\delta_0](x) \leq \frac{c_{N,\alpha}^p}{|x|^{(N-2\alpha)p}}, \quad \forall x \in \Omega \setminus \{0\},$$

then we apply (2.8) to obtain (2.9) and (2.10). \square

Let $\eta_0 : \mathbb{R}^N \rightarrow [0, 1]$ be a C^∞ radially symmetric function increasing with respect to $|x|$ such that $\eta_0 = 1$ in $\mathbb{R}^N \setminus B_2(0)$ and $\eta_0 = 0$ in $B_1(0)$. Let $\eta_\epsilon(x) = \eta_0(\epsilon^{-1}x)$ for $x \in \mathbb{R}^N$ and

$$u_\epsilon = u\eta_\epsilon. \quad (2.11)$$

By direct computation, we have that

$$\begin{aligned} (-\Delta)^\alpha u_\epsilon(x) &= \eta_\epsilon(x)(-\Delta)^\alpha u(x) + u(x)(-\Delta)^\alpha \eta_\epsilon(x) \\ &\quad - \frac{c_{N,\alpha}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta_\epsilon(x) - \eta_\epsilon(y))}{|x - y|^{N+2\alpha}} dy, \quad \forall x \in \Omega \setminus \{0\} \end{aligned}$$

and

$$(-\Delta)^\alpha u_\epsilon(0) = \lim_{x \rightarrow 0} (-\Delta)^\alpha u_\epsilon(x) = -c_{N,\alpha} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(y)\eta_\epsilon(y)}{|y|^{N+2\alpha}} dy.$$

Denote by L the operator related to $(-\Delta)^\alpha u - u^p$ in the distribution sense, i.e.

$$L(\xi) = \int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx, \quad \forall \xi \in C_c^\infty(\Omega). \quad (2.12)$$

Lemma 2.2. *For any $\xi \in C_c^\infty(\Omega)$ with the support in $\Omega \setminus \{0\}$,*

$$L(\xi) = 0.$$

Proof. For any $\xi \in C_c^\infty(\Omega)$, applying the Integral by Parts formula, see Lemma 2.2 in [16], it infers that

$$\int_{\Omega} \xi(-\Delta)^\alpha u_\epsilon dx = \int_{\Omega} u_\epsilon(-\Delta)^\alpha \xi dx. \quad (2.13)$$

Since $\xi \in C_c^\infty(\Omega)$ has the support in $\Omega \setminus \{0\}$, then there exists $r > 0$ such that $\xi = 0$ in $B_r(0)$ and if we put the $\epsilon > 0$ small enough, we have that

$$\begin{aligned} & \left| \int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx \right| \\ & \leq \left| \int_{\Omega} [u_\epsilon(-\Delta)^\alpha \xi - u^p \xi] dx \right| + \left| \int_{\Omega} |(1 - \eta_\epsilon)u(-\Delta)^\alpha \xi| dx \right| \\ & \leq \int_{B_{2\epsilon}(0)} |u(-\Delta)^\alpha \xi| dx + \int_{\Omega} |\xi(-\Delta)^\alpha u_\epsilon - u^p \xi| dx \\ & \quad + \int_{\Omega} u|\xi| |(-\Delta)^\alpha \eta_\epsilon| dx + \frac{c_{N,\alpha}}{2} \int_{\Omega} \left| \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta_\epsilon(x) - \eta_\epsilon(y))}{|x - y|^{N+2\alpha}} dy \right| |\xi(x)| dx \\ & = \int_{B_{2\epsilon}(0)} |u(-\Delta)^\alpha \xi| dx + \int_{\Omega \setminus B_r(0)} |\xi(-\Delta)^\alpha u_\epsilon - u^p \xi| dx + \int_{\Omega \setminus B_r(0)} u|\xi| |(-\Delta)^\alpha \eta_\epsilon(x)| dx \\ & \quad + \frac{c_{N,\alpha}}{2} \int_{\Omega \setminus B_r(0)} \left| \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta_\epsilon(x) - \eta_\epsilon(y))}{|x - y|^{N+2\alpha}} dy \right| |\xi(x)| dx, \end{aligned}$$

where u_ϵ is defined in (2.11). For $x \in \Omega \setminus B_r(0)$ and $\epsilon < \frac{r}{4}$, we have that

$$\begin{aligned} (-\Delta)^\alpha u_\epsilon(x) &= c_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u_\epsilon(y)}{|x - y|^{N+2\alpha}} dy \\ &= (-\Delta)^\alpha u(x) + c_{N,\alpha} \int_{B_{2\epsilon}(0)} \frac{(1 - \eta_\epsilon(y))u(y)}{|x - y|^{N+2\alpha}} dy, \end{aligned}$$

where $|x - y| > r - 2\epsilon$ and then we have that

$$\lim_{\epsilon \rightarrow 0} \|(-\Delta)^\alpha u_\epsilon - (-\Delta)^\alpha u\|_{L^\infty(\Omega \setminus B_r(0))} = 0,$$

which implies that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(0)} |\xi(-\Delta)^\alpha u_\epsilon - u^p \xi| dx = 0. \quad (2.14)$$

For $x \in \Omega \setminus B_r(0)$ and $\epsilon < \frac{r}{4}$, we have that

$$|(-\Delta)^\alpha \eta_\epsilon(x)| = c_{N,\alpha} \int_{B_{2\epsilon}(0)} \frac{1 - \eta_\epsilon(y)}{|x - y|^{N+2\alpha}} dy \leq c\epsilon^N (r - 2\epsilon)^{-N-2\alpha},$$

then we imply that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(0)} u|\xi| |(-\Delta)^\alpha \eta_\epsilon(x)| dx = 0. \quad (2.15)$$

Finally, for $x \in \Omega \setminus B_r(0)$ and $\epsilon < \frac{r}{4}$, there holds that

$$\begin{aligned} & \int_{\Omega \setminus B_r(0)} \left| \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta_\epsilon(x) - \eta_\epsilon(y))}{|x - y|^{N+2\alpha}} dy \right| |\xi(x)| dx \\ &= \int_{\Omega \setminus B_r(0)} \left| \int_{B_{2\epsilon}(0)} \frac{(u(x) - u(y))(1 - \eta_\epsilon(y))}{|x - y|^{N+2\alpha}} dy \right| |\xi(x)| dx \\ &\leq (r - 2\epsilon)^{-N-2\alpha} \|\xi\|_{L^\infty(\Omega)} \left[2^N c_{N,\alpha} \epsilon^N \int_{\Omega} u(x) dx + \int_{B_{2\epsilon}(0)} u(y) dy \right] \\ &\leq (r - 2\epsilon)^{-N-2\alpha} \|\xi\|_{L^\infty(\Omega)} \left[2^N c_{N,\alpha} \epsilon^N \int_{\Omega} u(x) dx + c_5 \epsilon^{2\alpha} \right] \\ &\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

which, together with (2.14) and (2.15), implies that

$$\int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx = 0.$$

Therefore, $L(\xi) = 0$ for any $\xi \in C_c^\infty(\Omega)$ with the support in $\Omega \setminus \{0\}$. \square

3. ISOLATED SINGULARITIES

From (2.1), then $u \in L^1(\Omega)$ and for any $\xi \in C_c^\infty(\Omega)$,

$$\left| \int_{\Omega} u(-\Delta)^\alpha \xi dx \right| < +\infty,$$

so L is a bounded functional of $C_c^\infty(\Omega)$. From Lemma 2.2, for any $\xi \in C_c^\infty(\Omega)$ with the support in $\Omega \setminus \{0\}$, then

$$L(\xi) = 0.$$

This means that the support of L is a isolated set $\{0\}$ and by Theorem XXXV in [30] (see also Theorem 6.25 in [33]) it implies that

$$L = \sum_{|a|=0}^{\infty} k_a D^a \delta_0, \quad (3.1)$$

where $a = (a_1, \dots, a_N)$ is a multiple index with $a_i \in \mathbb{N}$, $|a| = \sum_{i=1}^N a_i$ and in particular, $D^0 \delta_0 = \delta_0$. Then we have that

$$L(\xi) = \int_{\Omega} [u(-\Delta)^\alpha \xi - u^p \xi] dx = \sum_{|a|=0}^{\infty} k_a D^a \xi(0) \quad \text{for } \forall \xi \in C_c^\infty(\Omega). \quad (3.2)$$

Proposition 3.1. *Assume that $p > 1$ and a is multiple index. Then*

$$k_a = 0 \quad \text{for any } |a| \geq 2\alpha. \quad (3.3)$$

In particular, if $\alpha \in (0, \frac{1}{2}]$, then there exists $k \geq 0$ such that

$$L = k\delta_0.$$

Proof. For any multiple index $a = (a_1, \dots, a_N)$, let ζ_a be a C^∞ function such that

$$\text{supp}(\zeta_a) \subset \overline{B_1(0)} \quad \text{and} \quad \zeta_a(x) = k_a \prod_{i=1}^N x_i^{a_i} \quad \text{for } x \in B_1(0). \quad (3.4)$$

Now we use the following test functions in (3.2),

$$\xi_\epsilon(x) := \zeta_a(\epsilon^{-1}x), \quad \forall x \in \mathbb{R}^N.$$

Observe that

$$\sum_{|a| \leq q} k_a D^a \xi_\epsilon(0) = \frac{k_a^2}{\epsilon^{|a|}} \prod_{i=1}^N a_i!,$$

where $a_i! = a_i \cdot (a_i - 1) \cdots 1 > 0$ and $a_i! = 1$ if $a_i = 0$.

Let $r > 0$ and then

$$\begin{aligned} \left| \int_{\Omega} u(-\Delta)^\alpha \xi_\epsilon dx \right| &= \frac{1}{\epsilon^{2\alpha}} \left| \int_{\Omega} u(x) (-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right) dx \right| \\ &\leq \frac{1}{\epsilon^{2\alpha}} \left[\int_{\Omega \setminus B_r(0)} u(x) |(-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right)| dx + \int_{B_r(0)} u(x) |(-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right)| dx \right]. \end{aligned}$$

Fix r , we see that

$$|(-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right)| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \quad \text{uniformly in } \Omega \setminus B_r(0),$$

then

$$\int_{\Omega \setminus B_r(0)} u(x) |(-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right)| dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Furthermore,

$$\begin{aligned} \int_{B_r(0)} u(x) |(-\Delta)^\alpha \zeta_a\left(\frac{1}{\epsilon}x\right)| dx &\leq \|(-\Delta)^\alpha \zeta_a\|_{L^\infty(\mathbb{R}^N)} \left(\int_{B_r(0)} u^p(x) dx \right)^{\frac{1}{p}} |B_r(0)|^{\frac{p-1}{p}} \\ &\leq c_6 \|(-\Delta)^\alpha \zeta_a\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^p(\Omega)} r^{\frac{p-1}{p}N} \\ &\rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

Therefore, we have that

$$\left| \int_{\Omega} u(-\Delta)^\alpha \xi_\epsilon dx \right| = \epsilon^{-2\alpha} o(1). \quad (3.5)$$

On the other side, we see that

$$\begin{aligned} \int_{\Omega} u^p \xi_\epsilon dx &= \int_{B_\epsilon(0)} u^p(x) \zeta_a\left(\frac{1}{\epsilon}x\right) dx \\ &\leq \|\zeta_a\|_{L^\infty(\Omega)} \int_{B_\epsilon(0)} u^p(x) dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

For $|a| \geq 2\alpha$, we have that

$$k_a^2 \leq c_7 \epsilon^{|a|} [\epsilon^{-2\alpha} o(1) + o(1)] \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

then we have $k_a = 0$ by arbitrary of ϵ in (3.4). Thus (3.3) holds true.

In the particular of $\alpha \in (0, \frac{1}{2}]$, since $|a| \in \mathbb{N}$ and $2\alpha \leq 1$, then we have that $k_a = 0$ for all $|a| \geq 1$. The proof ends. \square

From Proposition 3.1, it implies that for $\alpha \in (\frac{1}{2}, 1)$, the expression (3.1) reduces to

$$L = k\delta_0 + \sum_{i=1}^N k_i D_i \delta_0. \quad (3.6)$$

where $\langle D_i \delta_0, \xi \rangle = \frac{\partial \xi(0)}{\partial x_i}$. We observe that

$$G_\alpha(x, y) = \frac{c_{N,\alpha}}{|x-y|^{N-2\alpha}} - g(x, y),$$

where g is α -harmonic function such that $g(x, y) = \frac{c_{N,\alpha}}{|x-y|^{N-2\alpha}}$ if $x \in \mathbb{R}^N \setminus \Omega$ or $y \in \mathbb{R}^N \setminus \Omega$, then we have that

$$\mathbb{G}_\alpha[D_i \delta_0](x) = (N-2\alpha) \frac{x_i}{|x|^{N-2\alpha+2}} - \partial_{x_i} g(x, 0),$$

and

$$|\partial_{x_i} g(x, 0)| \leq c_8 \rho^{\alpha-1}(x),$$

where $c_8 > 0$ and $\rho(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$.

Proposition 3.2. *Assume that $p > 1$, $\alpha \in (\frac{1}{2}, 1)$, and $k_i \in \mathbb{N}$ is from (3.6). Then*

$$k_i = 0 \quad \text{for any } i = 1, \dots, N. \quad (3.7)$$

Proof. Since

$$\begin{aligned} \Gamma(x) &:= k\mathbb{G}_\alpha[\delta_0](x) + \sum_{i=1}^N k_i \mathbb{G}_\alpha[D_i \delta_0] \\ &= \frac{c_{N,\alpha} k}{|x|^{N-2\alpha}} + c_{N,\alpha} \sum_{i=1}^N k_i \frac{x_i}{|x|^{N-2\alpha+2}} + c_{N,\alpha} \sum_{i=1}^N \partial_{x_i} g(x, 0), \end{aligned}$$

where g is a bounded function, then Γ must change signs if $k_i \neq 0$ for some i . Now assume that there exists i such that $k_i \neq 0$. We observe that

$$u = \mathbb{G}_\alpha[u^p] + \Gamma \quad (3.8)$$

and there is $t \in (0, 1)$ such that

$$A_t := \{x \in \Omega \setminus \{0\} : k_i \frac{x}{|x|} \cdot e_i > t\} \neq \emptyset$$

and

$$\frac{1}{c_9} |x|^{2\alpha-N-1} < \Gamma(x) \leq c_9 |x|^{2\alpha-N-1} + c_9 \rho^{\alpha-1}(x), \quad x \in A_t,$$

where $c_9 > 1$. So if $p \geq \frac{N}{N-2\alpha+1}$, then $\Gamma^p \notin L^1(A_t)$ and $u^p \notin L^1(\Omega)$, which contradicts (2.1).

Now we only have to consider the case that $p < \frac{N}{N-2\alpha+1}$. In order to obtain the contradiction, we continue to estimate $\mathbb{G}_\alpha[u^p]$. Let

$$u_1 = \mathbb{G}_\alpha[u^p].$$

We infer from $u^p \in L^{s_0}(\Omega)$ with $s_0 = \frac{1}{2}[1 + \frac{1}{p} \frac{N}{N-2\alpha+1}] > 1$ and Proposition 2.2 that $u_1 \in L^{s_1 p}(\Omega)$ and $u_1^p \in L^{s_1}(\Omega)$ with

$$s_1 = \frac{1}{p} \frac{N}{N-2\alpha s_0} s_0.$$

By (3.8),

$$u^p \leq c_{10}(u_1^p + k^p |\Gamma|^p) \quad \text{in } \Omega, \quad (3.9)$$

where $c_{10} > 0$. By the definition of u_1 and (3.9), we obtain

$$u_1 \leq c_{10}(\mathbb{G}_\alpha[u_1^p] + k^p \mathbb{G}_\alpha[|\Gamma|^p]), \quad (3.10)$$

where

$$k^p \mathbb{G}_\alpha[|\Gamma|^p](x) \leq c_2 |x|^{(2\alpha-1-N)p+2\alpha}$$

and

$$(2\alpha - 1 - N)p + 2\alpha > 2\alpha - 1 - N.$$

If $s_1 > \frac{1}{2\alpha} Np$, by Proposition 2.2, $\mathbb{G}_\alpha[u_1^p] \in L^\infty(\Omega)$. Hence, we know from (3.10) that

$$u_1(x) \leq c_{11} |x|^{(2\alpha-1-N)p+2\alpha}, \quad \forall x \in \Omega \setminus \{0\}. \quad (3.11)$$

In (3.8), $c_{N,\alpha} \sum_{i=1}^N k_i \frac{x_i}{|x|^{N-2\alpha+2}}$ has negative singularity as $|\cdot|^{2\alpha-1-N}$ in

$$-A_t := \{x \in \Omega : -x \in A_t\},$$

then from (3.11) and (3.8) with $(2\alpha - 1 - N)p + 2\alpha > 2\alpha - 1 - N$, there exists some point $x_0 \in -A_t$ such that $u(x_0) < 0$, which is impossible since u is nonnegative solution of (1.1).

On the other hand, if $s_1 < \frac{1}{2\alpha} Np$, we proceed as above. Let

$$u_2 = \mathbb{G}_\alpha[u_1^p].$$

By Proposition 2.2, $u_2 \in L^{s_2 p}(\Omega)$, where

$$s_2 = \frac{1}{p} \frac{Ns_1}{Np - 2\alpha s_1} > \frac{N}{N - s_0} s_1 > \left(\frac{1}{p} \frac{N}{N - 2\alpha s_0} \right)^2 s_0.$$

Inductively, we define

$$s_m = \frac{1}{p} \frac{Ns_{m-1}}{Np - 2\alpha s_{m-1}} > \left(\frac{1}{p} \frac{N}{N - 2\alpha s_0} \right)^m s_0.$$

So there is $m_0 \in \mathbb{N}$ such that

$$s_{m_0} > \frac{1}{2\alpha} Np$$

and by Proposition 2.2 part (i), it infers that

$$u_{m_0} \in L^\infty(\Omega).$$

Therefore, (3.11) holds true, it infers that $u(x_0) < 0$ for some point $x_0 \in \Omega \setminus \{0\}$ and we obtain a contradiction with $u \geq 0$. \square

Proof of Theorem 1.1. From Proposition 3.1 and Proposition 3.2, there exists some $k \in \mathbb{R}$ such that u is a weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{aligned} \quad (3.12)$$

and then

$$u = \mathbb{G}_\alpha[u^p] + k\mathbb{G}_\alpha[\delta_0].$$

When $p \geq \frac{N}{N-2\alpha}$, if $k \neq 0$,

$$u^p(x) \geq k^p \mathbb{G}_\alpha[\delta_0]^p(x) \geq c_{N,\alpha}^p k^p |x|^{-(N-2\alpha)p} \geq c_{N,\alpha}^p k^p |x|^{-N}, \quad x \in \Omega \cap B_1(0) \setminus \{0\},$$

then $u^p \notin L^1(\Omega)$ and contradicts (2.1). Therefore, if $p \geq \frac{N}{N-2\alpha}$, we have that $k = 0$.

When $p \in (1, \frac{N}{N-2\alpha})$ and $k = 0$, then

$$u = \mathbb{G}_\alpha[u^p].$$

We infer from $u^p \in L^{t_0}(\Omega)$ with $t_0 = \frac{1}{2}[1 + \frac{1}{p}\frac{N}{N-2\alpha}] > 1$ and Proposition 2.2 that $u \in L^{t_1 p}(\Omega)$ and $u^p \in L^{t_1}(\Omega)$ with

$$t_1 = \frac{1}{p} \frac{N}{N - 2\alpha t_0} t_0.$$

If $t_1 > \frac{1}{2\alpha} Np$, by Proposition 2.2, $u \in L^\infty(\Omega)$ and then it could be improved that u is a classical solution of (1.8).

If $t_1 < \frac{1}{2} Np$, we proceed as above. By Proposition 2.2, $u \in L^{t_2 p}(\Omega)$, where

$$t_2 = \frac{1}{p} \frac{N t_1}{N p - 2\alpha t_1} > \frac{N}{N - t_0} t_1 > \left(\frac{1}{p} \frac{N}{N - 2\alpha t_0} \right)^2 t_0.$$

Inductively, we define

$$t_m = \frac{1}{p} \frac{N t_{m-1}}{N p - 2\alpha t_{m-1}} > \left(\frac{1}{p} \frac{N}{N - 2\alpha t_0} \right)^m t_0.$$

So there is $m_0 \in \mathbb{N}$ such that

$$t_{m_0} > \frac{1}{2\alpha} Np$$

and by Proposition 2.2 part (i),

$$u \in L^\infty(\Omega),$$

then it deduces that u is a classical solution of (1.8).

When $p \in (1, \frac{N}{N-2\alpha})$ and $k \neq 0$, from observations that

$$\lim_{x \rightarrow 0} \mathbb{G}_\alpha[\delta_0](x) |x|^{N-2\alpha} = c_{N,\alpha}$$

and

$$u = \mathbb{G}_\alpha[u^p] + k \mathbb{G}_\alpha[\delta_0], \quad (3.13)$$

we infer from $u^p \in L^{t_0}(\Omega)$ for any $t_0 \in (1, \frac{1}{p}\frac{N}{N-2\alpha})$, letting

$$u_1 = \mathbb{G}_\alpha[u^p],$$

then it follows from Proposition 2.2 that if $\frac{1}{p}\frac{N}{N-2\alpha} > \frac{Np}{2\alpha}$, $u_1 \in L^\infty(\Omega)$, then u has asymptotic behavior $c_{N,\alpha} k |x|^{2\alpha-N}$ and we are done. If not, $u_1 \in L^{t_1 p}(\Omega)$ and $u_1^p \in L^{t_1}(\Omega)$ with

$$t_1 = \frac{1}{p} \frac{N}{N - 2\alpha t_0} t_0.$$

By the Young's inequality,

$$u^p \leq c_{11} \left(u_1^p + |k|^p |x|^{p(2\alpha-N)} \right) \quad \text{in } \Omega \setminus \{0\}, \quad (3.14)$$

where $c_{11} > 0$. By the definition of u_1 and (3.14), we obtain

$$u_1 \leq c_{11} \left(\mathbb{G}_\alpha \left[u_1^p + |k|^p |x|^{p(2\alpha-N)} \right] \right), \quad (3.15)$$

where

$$k^p \mathbb{G}_\alpha[|\cdot|^{p(2\alpha-N)}](x) \leq c_{12} |x|^{(2\alpha-N)p+2\alpha}$$

and

$$(2\alpha - N)p + 2\alpha > 2\alpha - N.$$

If $t_1 > \frac{1}{2\alpha} Np$, by Proposition 2.2, $\mathbb{G}_\alpha[u_1^p] \in L^\infty(\Omega)$. Hence, we have that

$$u(x) \leq c_{13} \mathbb{G}_\alpha[u_1^p] + c_{13} |x|^{(2\alpha-N)p+2\alpha} + k \mathbb{G}_\alpha[\delta_0](x), \quad \forall x \in \Omega \setminus \{0\}. \quad (3.16)$$

Since $(2\alpha - N)p + 2\alpha > 2\alpha - N$, we deduce from (3.16) that

$$\lim_{x \rightarrow 0} u(x) |x|^{N-2\alpha} = c_{N,\alpha} k. \quad (3.17)$$

On the other hand, if $t_1 < \frac{1}{2\alpha}Np$, we proceed as above. Let

$$u_2 = \mathbb{G}_\alpha[u_1^p].$$

By Proposition 2.2, $u_2 \in L^{t_2 p}(\Omega)$, where

$$t_2 = \frac{1}{p} \frac{Ns_1}{Np - 2\alpha t_1} > \frac{N}{N - t_0} t_1 > \left(\frac{1}{p} \frac{N}{N - 2\alpha t_0} \right)^2 t_0.$$

Inductively, we define

$$t_m = \frac{1}{p} \frac{Nt_{m-1}}{Np - 2\alpha t_{m-1}} > \left(\frac{1}{p} \frac{N}{N - 2\alpha t_0} \right)^m t_0.$$

So there is $m_0 \in \mathbb{N}$ such that

$$t_{m_0} > \frac{1}{2\alpha}Np$$

and

$$u_{m_0} \in L^\infty(\Omega).$$

Therefore, by the assumption that u is nonnegative, it is necessary that $k > 0$ and

$$\lim_{x \rightarrow 0} u(x)|x|^{N-2\alpha} = c_{N,\alpha}k.$$

This ends the proof. \square

4. EXISTENCE OF WEAK SOLUTION

4.1. Minimal solution. *Proof of Existence of the minimal solution in Theorem 1.2.* We first define the iterating sequence

$$v_0 := k\mathbb{G}_\alpha[\delta_0] > 0,$$

and

$$v_n = \mathbb{G}_\alpha[v_{n-1}^p] + k\mathbb{G}_\alpha[\delta_0].$$

Observing that

$$v_1 = \mathbb{G}_\alpha[(kv_0)^p] + k\mathbb{G}_\alpha[\delta_0] > v_0$$

and assuming that

$$v_{n-1} \geq v_{n-2} \quad \text{in } \Omega \setminus \{0\},$$

we deduce that

$$v_n = \mathbb{G}_\alpha[v_{n-1}^p] + k\mathbb{G}_\alpha[\delta_0] \geq \mathbb{G}_\alpha[v_{n-2}^p] + k\mathbb{G}_\alpha[\delta_0] = v_{n-1}.$$

Thus, the sequence $\{v_n\}$ is a increasing with respect to n . Moreover, we have that

$$\int_\Omega v_n (-\Delta)^\alpha \xi \, dx = \int_\Omega v_{n-1}^p \xi \, dx + k\xi(0), \quad \forall \xi \in C_c^\infty(\Omega). \quad (4.1)$$

We next build an upper bound for the sequence $\{v_n\}$. For $t > 0$, denote

$$w_t = tk^p \mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] + k\mathbb{G}_\alpha[\delta_0] \leq (c_2 tk^p + k) \mathbb{G}_\alpha[\delta_0], \quad (4.2)$$

where $c_2 > 0$ is from Lemma 2.1, then

$$\mathbb{G}_\alpha[w_t^p] + k\mathbb{G}_\alpha[\delta_0] \leq (c_2 tk^p + k)^p \mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] + k\mathbb{G}_\alpha[\delta_0] \leq w_t,$$

if

$$(c_2 tk^p + k)^p \leq tk^p,$$

that is

$$(c_2 tk^{p-1} + 1)^p \leq t. \quad (4.3)$$

Note that the convex function $f_k(t) = (c_2 t k^{p-1} + 1)^p$ can intersect the line $g(t) = t$, if

$$c_2 k^{p-1} \leq \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1}. \quad (4.4)$$

Let $k_p = \left(\frac{1}{c_2 p} \right)^{\frac{1}{p-1}} \frac{p-1}{p}$, then if $k \leq k_p$, it always hold that $f_k(t_p) \leq t_p$ for $t_p = \left(\frac{p}{p-1} \right)^p$. Hence, for t_p we have chosen, by the definition of w_{t_p} , we have $w_{t_p} > v_0$ and

$$v_1 = \mathbb{G}_\alpha[v_0^p] + k \mathbb{G}_\alpha[\delta_0] < \mathbb{G}_\alpha[w_{t_p}^p] + k \mathbb{G}_\alpha[\delta_0] = w_{t_p}.$$

Inductively, we obtain

$$v_n \leq w_{t_p} \quad (4.5)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{v_n\}$ converges. Let $u_k := \lim_{n \rightarrow \infty} v_n$. By (4.1), u_k is a weak solution of (1.6).

We claim that u_k is the minimal solution of (1.1), that is, for any positive solution u of (1.6), we always have $u_k \leq u$. Indeed, there holds

$$u = \mathbb{G}_\alpha[u^p] + k \mathbb{G}_\alpha[\delta_0] \geq v_0,$$

and then

$$u = \mathbb{G}_\alpha[u^p] + k \mathbb{G}_\alpha[\delta_0] \geq \mathbb{G}_\alpha[v_0^p] + k \mathbb{G}_\alpha[\delta_0] = v_1.$$

We may show inductively that

$$u \geq v_n$$

for all $n \in \mathbb{N}$. The claim follows.

Similarly, if problem (1.6) has a nonnegative solution u for $k_1 > 0$, then (1.6) admits a minimal solution u_k for all $k \in (0, k_1]$. As a result, the mapping $k \mapsto u_k$ is increasing. So we may define

$$k^* = \sup\{k > 0 : (1.1) \text{ has minimal solution for } k\},$$

then k^* is the largest k such that problem (1.6) has minimal positive solution, and

$$k^* \geq k_p.$$

We next prove that $\lambda^* < +\infty$. Let (λ_1, φ_1) be the first eigenvalue and positive eigenfunction of $(-\Delta)^\alpha$ in $H_0^\alpha(\Omega)$, see Proposition 5 in [31] then by the fact that

$$u_k \geq k \mathbb{G}_\alpha[\delta_0] \geq c_{14} k \quad \text{in } B_r(0)$$

for some $r > 0$ satisfying $B_{2r}(0) \subset \Omega$. There exists $c_{14} > 1$ such that

$$c_{14} \int_{B_r(0)} u_k \varphi_1 dx \geq \int_{\Omega} u_k \varphi_1 dx$$

and we imply that

$$\begin{aligned} c_{14} \lambda_1 \int_{B_r(0)} u_k \varphi_1 dx &\geq \lambda_1 \int_{\Omega} u_k \varphi_1 dx = \int_{\Omega} u_k (-\Delta)^\alpha \varphi_1 dx \\ &\geq \int_{\Omega} u_k^p \varphi_1 dx + k \varphi_1(x_0) \geq c_{15} k^{p-1} \int_{B_r(0)} u_k \varphi_1 dx, \end{aligned}$$

then k must satisfy

$$k \leq c_{16} \lambda_1^{-\frac{1}{p-1}}.$$

where $c_{15}, c_{16} > 0$. As a conclusion, there exists $c_{17} > 0$ such that

$$k^* \leq c_{17} \lambda_1^{-\frac{1}{p-1}}. \quad (4.6)$$

Regularity of very weak solution of (1.6). Let u be a very weak solution of (1.6) and $x_0 \in \Omega \setminus \{0\}$, then

$$\begin{aligned} u &= \mathbb{G}_\alpha[u^p] + k\mathbb{G}_\alpha[\delta_0] \\ &= \mathbb{G}_\alpha[u^p \chi_{B_r(x_0)}] + \mathbb{G}_\alpha[u^p \chi_{\Omega \setminus B_r(x_0)}] + k\mathbb{G}_\alpha[\delta_0] \end{aligned}$$

where $\mathbb{G}_\alpha[\delta_0]$ is $C_{loc}^\infty(\Omega \setminus \{0\})$, $r > 0$ such that $\overline{B_{2r}(0)} \subset \Omega \setminus \{0\}$. Let $B_i = B_{2^{-i}r}(x_0)$. For $x \in B_i$, we have that

$$\mathbb{G}_\alpha[\chi_{\Omega \setminus B_{i-1}} u^p](x) = \int_{\Omega \setminus B_{i-1}} u^p(y) G_\alpha(x, y) dy,$$

then, for some $C_i > 0$, we have

$$\|\mathbb{G}_\alpha[\chi_{\Omega \setminus B_i} u^p]\|_{C^2(B_{i-1})} \leq C_i \|u^p\|_{L^1(B_{2r}(x_0))} \quad (4.7)$$

and for some constant $c_i > 0$ depending on i , we have

$$\|\mathbb{G}_\alpha[\delta_0]\|_{C^2(B_{i-1})} \leq c_i |x_0|^{2-N}. \quad (4.8)$$

By Proposition 2.2, $u^p \in L^{q_0}(B_{2r_0}(x_0))$ with $q_0 = \frac{1}{2}(1 + \frac{1}{p} \frac{N}{N-2\alpha}) > 1$. By Proposition 2.2 again we find

$$\mathbb{G}_\alpha[\chi_{B_{2r}(x_0)} u^p] \in L^{p_1}(B_{2r}(x_0)) \text{ with } p_1 = \frac{Nq_0}{N-2\alpha q_0}.$$

Similarly,

$$u^p \in L^{q_1}(B_r(x_0)) \text{ with } q_1 = \frac{p_1}{p},$$

and

$$\mathbb{G}_\alpha[\chi_{B_r(x_0)} u^p] \in L^{p_2}(B_r(x_0)) \text{ with } p_2 = \frac{Nq_1}{N-2\alpha q_1}.$$

Let $q_i = \frac{p_i}{p}$ and $p_{i+1} = \frac{Nq_i}{N-2\alpha q_i}$ if $N-2q_i > 0$. Then we obtain inductively that

$$u^p \in L^{q_i}(B_i) \quad \text{and} \quad \mathbb{G}[\chi_{B_i} u^p] \in L^{p_{i+1}}(B_i).$$

We may verify that

$$\frac{q_{i+1}}{q_i} = \frac{1}{p} \frac{N}{N-2\alpha q_i} > \frac{1}{p} \frac{N}{N-2\alpha q_1} > 1.$$

Therefore, $\lim_{i \rightarrow +\infty} q_i = +\infty$, so there exists i_0 such that $N-2q_{i_0} > 0$, but $N-2q_{i_0+1} < 0$, and we deduce that

$$\mathbb{G}_\alpha[\chi_{B_{i_0}} u^p] \in L^\infty(B_{i_0}).$$

As a result,

$$u \in L^\infty(B_{i_0}).$$

By elliptic regularity, we know from (4.8) that u is Hölder continuous in B_{i_0} and so is u^p . Hence, u is a classical solution of (1.1). \square

4.2. Stability. In what follows, we discuss the stability of the minimal solution of (1.1).

Definition 4.1. A solution (or weak solution) u of (1.1) is stable (resp. semi-stable) if

$$\|\xi\|_\alpha^2 > p \int_\Omega u^{p-1} \xi^2 dx, \quad (\text{resp. } \geq) \quad \forall \xi \in H_0^\alpha(\Omega) \setminus \{0\}.$$

Proposition 4.1. For $k \in (0, k^*)$, let u_k be the minimal positive solution of (1.1) by Theorem 1.1. Then u_k is stable.

Moreover, there exists $c_{18} > 0$ such that for any $\xi \in H_0^\alpha(\Omega) \setminus \{0\}$,

$$\|\xi\|_\alpha^2 - p \int_\Omega u_k^{p-1} \xi^2 dx \geq c_{18} \left((k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}} \right) \|\xi\|_\alpha^2. \quad (4.9)$$

Proof. To prove the stability when $k > 0$ small. When $k > 0$ small, the iteration procedure: $v_n = \mathbb{G}_\alpha[v_{n-1}^p] + k\mathbb{G}_\alpha[\delta_0]$ is controlled by super solution w_{t_p} , where

$$w_{t_p} = t_p k^p \mathbb{G}_\alpha[\mathbb{G}_\alpha^p[\delta_0]] + k\mathbb{G}_\alpha[\delta_0],$$

then $u_k \leq w_{t_p}$ and there exists $c_{19} > 0$ such that

$$u_k(x) \leq c_{19} k |x|^{2\alpha-N}, \quad \forall x \in \Omega \setminus \{0\}.$$

So there holds

$$u_k^{p-1}(x) \leq c_{20} k^{p-1} |x|^{(2\alpha-N)(p-1)}.$$

Then it follows by Proposition 2.3 that

$$\int_{\Omega} u_k^{p-1} \xi^2 dx \leq c_{20} k^{p-1} \int_{\Omega} \frac{\xi_n^2(x)}{|x|^{(N-2\alpha)(p-1)}} dx < \frac{1}{p} \|\xi\|_{\alpha}^2$$

if $k > 0$ sufficient small. Then u_k is a stable solution of (1.1) for $k > 0$ small.

Now we prove the stability for $k \in (0, k^*)$. Suppose that if u_k is not stable, then we have that

$$\sigma_1 := \inf_{\xi \in H_0^\alpha(\Omega) \setminus \{0\}} \frac{\|\xi\|_{\alpha}^2}{p \int_{\Omega} u_k^{p-1} \xi^2 dx} \leq 1. \quad (4.10)$$

By Proposition 2.3, σ_1 is achievable and its achieved function ξ_1 could be setting to be nonnegative and satisfies

$$(-\Delta)^\alpha \xi_1 = \sigma_1 p u_k^{p-1} \xi_1.$$

Choosing $\hat{k} \in (k, k^*)$ and letting $w = u_{\hat{k}} - u_k > 0$, then we have that

$$w = \mathbb{G}_\alpha[u_{\hat{k}}^p - u_k^p] + (\hat{k} - k)\mathbb{G}_\alpha[\delta_0].$$

By the elementary inequality

$$(a+b)^p \geq a^p + p a^{p-1} b \quad \text{for } a, b \geq 0,$$

we infers that

$$w \geq \mathbb{G}_\alpha[p u_k^{p-1} w] + (\hat{k} - k)\mathbb{G}_\alpha[\delta_0].$$

Then

$$\begin{aligned} \sigma_1 \int_{\Omega} p u_k^{p-1} w \xi_1 dx &= \int_{\Omega} \xi_1 (-\Delta)^\alpha w dx \\ &\geq \int_{\Omega} p u_k^{p-1} w \xi_1 dx + (\hat{k} - k) \xi_1(0) > \int_{\Omega} p u_k^{p-1} w \xi_1 dx, \end{aligned}$$

which is impossible. Consequently,

$$p \int_{\Omega} u_k^{p-1} \xi^2 dx > \|\xi\|_{\alpha}^2, \quad \forall \xi \in H_0^\alpha(\Omega).$$

As a conclusion, we derive that u_k is stable for $k < k^*$.

To prove (4.9). For any $k \in (0, k^*)$, let $k' = \frac{k+k^*}{2} > k$ and $l_0 = (\frac{k}{k'})^{\frac{1}{p}} < 1$, then we see that the minimal solution $u_{k'}$ of (1.1) with k' is stable and

$$\begin{aligned} l_0 u_{k'} &\geq l_0^p u_{k'} \\ &= l_0^p (\mathbb{G}_\alpha[u_{k'}^p] + k' \mathbb{G}_\alpha[\delta_0]) + (k - k' l_0^p) \left\{ \mathbb{G}_\alpha\left[\frac{u_{k'}}{|x|^2}\right] + \mathbb{G}_\alpha[\delta_0] \right\} \\ &= \mathbb{G}_\alpha[(l_0 u_{k'})^p] + \mathbb{G}_\alpha\left[\frac{l_0 u_{k'}}{|x|^2}\right] + k \mathbb{G}_\alpha[\delta_0], \end{aligned}$$

where we have used $k - k'l_0^p = 0$. Thus, we have that $l_0 u_{k'}$ is the minimal solution of (1.1) and we have that

$$l_0 u_{k'} \geq u_k,$$

so for $\xi \in H_0^\alpha(\Omega) \setminus \{0\}$, we have that

$$\begin{aligned} 0 < \|\xi\|_\alpha^2 - p \int_\Omega u_{k'}^{p-1} \xi^2 dx &\leq \|\xi\|_\alpha^2 - p l_0^{1-p} \int_\Omega u_k^{p-1} \xi^2 dx \\ &= l_0^{1-p} \left[l_0^{p-1} \|\xi\|_\alpha^2 - p \int_\Omega u_k^{p-1} \xi^2 dx \right], \end{aligned}$$

thus,

$$\begin{aligned} \|\xi\|_\alpha^2 - p \int_\Omega u_k^{p-1} \xi^2 dx &= (1 - l_0^{p-1}) \|\xi\|_\alpha^2 + \left[l_0^{p-1} \|\xi\|_\alpha^2 - p \int_\Omega u_k^{p-1} \xi^2 dx \right] \\ &\geq (1 - l_0^{p-1}) \|\xi\|_\alpha^2, \end{aligned}$$

which together with the fact that

$$1 - l_0^{p-1} \geq c_{21} [(k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}],$$

implies (4.9). \square

4.3. Extremal solution. We would like to approach the weak solution when $k = k^*$ by the minimal solution u_k with $k < k^*$.

Proof of Theorem 1.2 in the case of $k = k^$.* Let (λ_1, φ_1) be the first eigenvalue and positive eigenfunction of $(-\Delta)^\alpha$ in $H_0^\alpha(\Omega)$, then for $k \in (0, k^*)$

$$\begin{aligned} \int_\Omega u_k^p \varphi_1 dx &= \int_\Omega u_k (-\Delta)^\alpha \varphi_1 dx - k \varphi_1(0) \\ &< \lambda_1 \left(\int_\Omega u_k^p \varphi_1 dx \right)^{\frac{1}{p}} \left(\int_\Omega \varphi_1 dx \right)^{1-\frac{1}{p}} \end{aligned}$$

which implies that

$$\|u_k\|_{L^p(\Omega, \rho^\alpha dx)} \leq \lambda_1^{\frac{p}{p-1}} \int_\Omega \varphi_1 dx.$$

Combine the mapping $k \mapsto u_k$ is increasing, then $u_{k^*} = \lim_{k \nearrow k^*} u_k$ exists and $u_k \rightarrow u_{k^*}$ in $L^p(\Omega, \rho^\alpha dx)$, thus,

$$\int_\Omega u_{k^*} (-\Delta)^\alpha \xi dx = \int_\Omega u_{k^*}^p \xi dx + k^* \xi(0), \quad \forall \xi \in C_c^\infty(\Omega).$$

So we conclude that (1.1) has a weak solution and then (1.1) has minimal solution u_{k^*} .

To prove that u_{k^} is semi-stable.* For any $\epsilon > 0$ and $\xi \in H_0^\alpha(\Omega) \setminus \{0\}$, there exists $k(\epsilon) > 0$ such that for all $k \in (k(\epsilon), k^*)$,

$$p \int_\Omega u_{k^*}^p \xi dx \leq p \int_\Omega u_k^p \xi dx + (k^* - k) p \int_\Omega u_{k^*}^p \xi dx \leq \|\xi\|_\alpha^2 + \epsilon$$

By the arbitrary of ϵ , we have that u_{k^*} is semi-stable.

To prove the uniqueness. If problem (1.1) admits a solution $u > u_{k^*}$.

$$\sigma_1 := \inf_{\xi \in H_0^\alpha(\Omega) \setminus \{0\}} \frac{\|\xi\|_\alpha^2}{p \int_\Omega u_{k^*}^{p-1} \xi^2 dx} \geq 1.$$

By the compact embedding theorem, σ_1 is achievable and its achieved function ξ_1 could be setting to be nonnegative and satisfies

$$(-\Delta)^\alpha \xi_1 = \sigma_1 p u_{k^*}^{p-1} \xi_1 \quad \text{in } \Omega, \quad \xi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Letting $w = u - u_{k^*} > 0$, then we have that

$$w = \mathbb{G}_\alpha[u_{k^*}^p - w^p].$$

By the elementary inequality

$$(a + b)^p > a^p + pa^{p-1}b \quad \text{for } a > b > 0,$$

we infer that

$$w > \mathbb{G}_\alpha[pu_{k^*}^{p-1}w],$$

then

$$\sigma_1 \int_\Omega pu_{k^*}^{p-1}w\xi_1 dx = \int_\Omega (-\Delta)w\xi_1 dx > \int_\Omega pu_{k^*}^{p-1}w\xi_1 dx$$

which is impossible with $\sigma_1 \leq 1$. As a conclusion, u_{k^*} is the unique solution of (1.1) with $k = k^*$. \square

4.4. Mountain-Pass type solution. For the second solution of (1.1), we would like to apply the Mountain-Pass theorem to find a positive weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= (u_k + u_+)^p - u_k^p & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \quad (4.11)$$

where $k \in (0, k^*)$ and u_k is the minimal positive solution of (1.1) obtained by Theorem 1.1. The second solution of (1.1) is derived by following proposition.

Proposition 4.2. *Assume that $p \in (1, \frac{N}{N-2\alpha})$, $k \in (0, k^*)$ and u_k is the minimal positive solution of (1.1) obtained by Theorem 1.1.*

Then problem (4.11) has a positive solution $v_k > u_k$.

Proof. We would like to employ the Mountain Pass theorem to look for the weak solution of (4.11). A function v is said to be a weak solution of (4.11) if

$$\langle u, \xi \rangle_\alpha = \int_\Omega [(u_k + u_+)^p - u_k^p] \xi dx, \quad \forall \xi \in H_0^\alpha(\Omega). \quad (4.12)$$

The natural functional associated to (4.11) is the following

$$E(v) = \frac{1}{2} \|v\|_\alpha^2 - \int_\Omega F(u_k, v_+) dx, \quad \forall v \in H_0^\alpha(\Omega), \quad (4.13)$$

where

$$F(s, t) = \frac{1}{p+1} [(s + t_+)^{p+1} - s^{p+1} - (p+1)s^p t_+]. \quad (4.14)$$

We observe that for any $\epsilon > 0$, there exists some $c_\epsilon > 0$, depending only on p , such that

$$0 \leq F(s, t) \leq (p + \epsilon)s^{p-1}t^2 + c_\epsilon t^{p+1}, \quad s, t \geq 0$$

By we have that for any $v \in H_0^\alpha(\Omega)$,

$$\begin{aligned} \int_\Omega F(u_k, v_+) dx &\leq (p + \epsilon) \int_\Omega u_k^{p-1} v_+^2 dx + c_\epsilon \int_\Omega v_+^{p+1} dx \\ &\leq c_{22} \|v\|_\alpha^2, \end{aligned}$$

then E is well-defined in $H_0^\alpha(\Omega)$.

We observe that $E(0) = 0$ and let $v \in H_0^\alpha(\Omega)$ with $\|v\|_\alpha = 1$, then for $k \in (0, k^*)$, choosing $\epsilon > 0$ small enough, it infers from (4.9) that

$$\begin{aligned} E(tv) &= \frac{1}{2}t^2\|v\|_\alpha^2 - \int_\Omega F(u_k, tv_+) \, dx \\ &\geq t^2 \left(\frac{1}{2}\|v\|_\alpha^2 - (p + \epsilon) \int_\Omega v_k^{p-1} v^2 \, dx \right) - c_{23}t^{p+1} \int_\Omega |v|^{p+1} \, dx \\ &\geq c_{24}t^2\|v\|_\alpha^2 - c_{23}t^{p+1}\|v\|_\alpha^{p+1} \\ &\geq \frac{c_{24}}{2}t^2 - c_{23}t^{p+1}, \end{aligned}$$

where $c_{23}, c_{24} > 0$ depend on k, k^* and we used (2.12) in the first inequality. So there exists $\sigma_0 > 0$ small, then for $\|v\|_{H_0^\alpha(\Omega)} = 1$, we have

$$E(\sigma_0 v) \geq \frac{c_{24}}{4}\sigma_0^2 =: \beta > 0.$$

We take a nonnegative function $v_0 \in H_0^\alpha(\Omega)$ and then

$$F(u_k, tv_0) \geq \frac{1}{p+1}t^{p+1}v_0^{p+1} - tu_k^p v_0.$$

Since the space of $\{tv_0 : t \in \mathbb{R}\}$ is a subspace of $H_0^\alpha(\Omega)$ with dimension 1 and all the norms are equivalent, then $\int_\Omega V_0 v_0(x)^{p+1} \, dx > 0$. Then there exists $t_0 > 0$ such that for $t \geq t_0$,

$$\begin{aligned} E(tv_0) &= \frac{t^2}{2}\|v_0\|_\alpha^2 - \int_\Omega F(u_k, tv_0) \, dx \\ &\leq \frac{t^2}{2}\|v_0\|_\alpha^2 - c_{24}t^{p+1} \int_\Omega v_0^{p+1} \, dx + t \int_\Omega u_k^p v_0 \, dx \\ &\leq c_{25}(t^2 + t - t^{p+1}) \leq 0, \end{aligned}$$

where $c_{24}, c_{25} > 0$. We choose $e = t_0 v_0$, we have $E(e) \leq 0$.

We next prove that E satisfies (PS) condition. We say that E has PS condition, if for any sequence $\{v_n\}$ in $H_0^\alpha(\Omega)$ satisfying $E(v_n) \rightarrow c$ and $E'(v_n) \rightarrow 0$ as $n \rightarrow \infty$, there is a convergent subsequence. Here the energy level c of functional E is characterized by

$$c = \inf_{\gamma \in \Upsilon} \max_{s \in [0,1]} E(\gamma(s)), \quad (4.15)$$

where $\Upsilon = \{\gamma \in C([0,1] : H_0^\alpha(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}$. We observe that

$$c \geq \beta.$$

Let $\{v_n\}$ in $H_0^\alpha(\Omega)$ satisfying $E(v_n) \rightarrow c$ and $E'(v_n) \rightarrow 0$ as $n \rightarrow \infty$, then we only have to show that there are a subsequence, still denote it by $\{v_n\}$ and $v \in H_0^\alpha(\Omega)$ such that

$$v_n \rightarrow v \quad \text{in } L^2(\Omega, u_k^{p-1} \, dx) \quad \text{and } L^{p+1}(\Omega) \quad \text{as } n \rightarrow \infty.$$

For some $c_{21} > 0$, we have that

$$c_{21}\|w\|_\alpha \geq E'(v_n)w = \langle v_n, w \rangle_\alpha - \int_\Omega f(u_k, (v_n)_+) w \, dx$$

and

$$c + 1 \geq E(v_n) = \frac{1}{2}\|v_n\|_\alpha^2 - \int_\Omega F(u_k, (v_n)_+) \, dx. \quad (4.16)$$

Let $c_p = \min\{1, p-1\}$, then it follows by [24, C.2 (iv)] that

$$f(s, t)t - (2 + c_p)F(s, t) \geq -\frac{c_p p}{2}s^{p-1}t^2, \quad s, t \geq 0,$$

thus $(2 + c_p) \times (4.16) - \langle E'(v_n), (v_n)_+ \rangle$ implies that

$$\begin{aligned} c + c_{21} \|v_n\|_\alpha &\geq \frac{c_p}{2} \|v_n\|_\alpha^2 - \int_\Omega [(2 + c_p)F(u_k, (v_n)_+) - f(u_k, (v_n)_+)(v_n)_+] dx \\ &\geq \frac{c_p}{2} \left[\|v_n\|_\alpha^2 - p \int_\Omega u_k^{p-1} v_n^2 dx \right] \\ &\geq c_{26} \frac{c_p}{2} \|v_n\|_\alpha^2, \end{aligned}$$

where $c_{26} > 0$. Therefore, we derive that v_n is uniformly bounded in $H_0^\alpha(\Omega)$ for $k \in (0, k^*)$.

Thus there exists a subsequence $\{v_n\}$ and v such that

$$v_n \rightharpoonup v \quad \text{in } H_0^\alpha(\Omega),$$

$$v_n \rightarrow v \quad \text{a.e. in } \Omega \quad \text{and in } L^{p+1}(\Omega), \quad L^2(\Omega, u_k^{p-1} dx),$$

when $n \rightarrow \infty$. Here we have used that $u_k^{p-1} \leq c_{27}|x|^{(2\alpha-N)(p-1)}$, where $(2\alpha - N)(p - 1) > -2\alpha$ and by Proposition 2.3 the embedding: $H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega, |x|^{(2\alpha-N)(p-1)} dx)$ is compact for $q \in [1, \frac{2N+2(2\alpha-N)(p-1)}{N-2\alpha})$, particularly, for $q = 2$.

We observe that

$$\begin{aligned} &|F(u_k, v_n) - F(u_k, v)| \\ &= \frac{1}{p+1} |(u_k + (v_n)_+)^p - (u_k + v_+)^p - (p+1)u_k^p((v_n)_+ - v_+)| \\ &\leq c_{28}u_k^{p-1}((v_n)_+ - v_+)^2 + c_{28}((v_n)_+ - v_+)^{p+1}, \end{aligned}$$

which implies that

$$F(u_k, v_n) \rightarrow F(u_k, v) \quad \text{a.e. in } \Omega \quad \text{and in } L^1(\Omega).$$

Then, together with $\lim_{n \rightarrow \infty} E(v_n) = c$, we have that $\|v_n\|_\alpha \rightarrow \|v\|_\alpha$ as $n \rightarrow \infty$. Then we obtain that $v_n \rightarrow v$ in $H_0^\alpha(\Omega)$ as $n \rightarrow \infty$.

Now Mountain Pass Theorem (for instance, [29, Theorem 6.1]; see also [28]) is applied to obtain that there exists a critical point $v \in H_0^\alpha(\Omega)$ of E at some value $c \geq \beta > 0$. By $\beta > 0$, we have that v is nontrivial and nonnegative. Then v is a positive weak solution of v of (4.11). By using bootstrap argument in [21], the interior regularity of v could be improved to be in $H_0^\alpha(\Omega) \cap C^2(\Omega \setminus \{0\})$, since u_k is locally bounded in $\Omega \setminus \{0\}$ and $p < \frac{N}{N-2\alpha}$. Since

$$u_k^{p-1}(x) \leq c_{27}|x|^{(N-2\alpha)(p-1)}, \quad \forall x \in B_1(0),$$

with $p < \frac{N}{N-2\alpha}$, we have that there is some $q > \frac{N}{2\alpha}$ such that

$$u_k^{p-1} \in L^q(B_1(0)),$$

so v_k is bounded at the origin. Moreover, by Maximum Principle, we conclude that $v > 0$ in Ω . \square

Proof of the existence Mountain-Pass type solution in Theorem 1.2. From Proposition 4.2, we obtain that there is a positive weak solution of v_k of (4.11), then v_k is weak solution of (4.11) and it holds that

$$\int_\Omega v_k(-\Delta)^\alpha \xi dx = \int_\Omega [(u_k + v_k)^p - u_k^p] \xi dx, \quad \forall \xi \in C_c^\infty(\Omega).$$

then $(u_k + v_k)$ satisfies

$$\int_\Omega (u_k + v_k)(-\Delta)^\alpha \xi dx = \int_\Omega (u_k + v_k)^p \xi dx + k\xi(0), \quad \forall \xi \in C_c^\infty(\Omega).$$

This means that $v_k + u_k$ is weak solution of (1.6) such that $v_k + u_k > u_k$ and $v_k + u_k$ to C^2 locally in $\Omega \setminus \{0\}$. \square

REFERENCES

- [1] P. Aviles, Local behaviour of the solutions of some elliptic equations, *Comm. Math. Phys.* 108, 177-192 (1987).
- [2] P. Baras and M. Pierre, Singularité séliminables pour des équations semi linéaires, *Ann. Inst. Fourier Grenoble* 34, 185-206 (1984).
- [3] P. Baras and M. Pierre, Critere d'existence de soltuions positive pour des equations semi-lineaires non monotones, *Ann. Inst. H. Poincaré Ana. Non lineaire* 2, 185-212 (1985).
- [4] H. Brezis and P. Lions, A note on isolated singularities for linear elliptic equations, in Mathematical Analysis and Applications, *Acad. Press*, 263-266 (1981).
- [5] Ph. Bénilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, *J. Evolution Eq.* 3, 673-770, (2003).
- [6] Ph. Bénilan, H. Brezis and M. Crandall, A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* 2, 523-555 (1975).
- [7] M. F. Bidaut-Véron and L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case, *Rev. Mat. Iberoamericana* 16, 477-513 (2000).
- [8] H. Brezis, Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems, *Proc. Internat. School, Erice, Wiley, Chichester*, 53-73 (1980).
- [9] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42, 271-297 (1989).
- [10] L. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian, *Invent. math.* 171, 425-461 (2008).
- [11] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equaitons, *Comm. Pure Appl. Math.* 62, 597-638 (2009).
- [12] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, *Arch. Ration. Mech. Anal.* 200, 59-88 (2011).
- [13] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, accepted by *Ann. Inst. H. Poincaré, Analyse Non Linéaire*, DOI: 10.1016/j.anihpc.2014.08.001.
- [14] L. Caffarelli and L. Silvestre, The Evans-Krylov theorem for non local fully non linear equations. *Ann. Math.* 174, 1163-1187 (2011).
- [15] H. Chen, P. Felmer and L. Véron, Elliptic equations involving general subcritical source nonlinearity and measures, arXiv:1409.3067.
- [16] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *J. Differential Equations* 257(5), 1457-1486 (2014).
- [17] H. Chen and L. Véron, Semilinear fractional elliptic equations with gradient nonlinearity involving measures, *J. Funct. Anal.* 266(8), 5467-5492 (2014).
- [18] Z. Chen, and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann.* 312, 465-501 (1998).
- [19] B. Gidas and J. Spruck, Global and local behaviour of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 34, 525-598 (1981).
- [20] A. Gmira and L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations, *Duke Math. J.* 64, 271-324 (1991).
- [21] Q. Han and F. Lin, Elliptic partial differential equations, *American Mathematical Soc.* 1, (2011).
- [22] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, *Arch. Rat. Mech. Anal.* 144, 201-231 (1998).
- [23] M. Marcus and L. Véron, Removable singularities and boundary traces, *J. Math. Pures Appl.* 80, 879-900 (2001).
- [24] Y. Naito and T. Sato, Positive solutions for semilinear elliptic equations with singular forcing terms, *J. Differential Equations* 235, 439-483 (2007).
- [25] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136, 521-573 (2012).
- [26] P. Lions, Isolated singularities in semilinear problems, *J. Differential Equations* 38(3), 441-450 (1980).
- [27] R. Cignoli and M. Cottlar, An Introduction to Functional Analysis, *North-Holland, Amsterdam* (1974).

- [28] P. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.* 65, American Mathematical Society (1986).
- [29] M. Struwe, Variational methods, applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, Springer Verlag (1990).
- [30] L. Schwartz, Theorie des distributions, *Hermann, Paris* (1966).
- [31] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* 33, 2105-2137 (2013).
- [32] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, *Princeton University Press* (1970).
- [33] W. Rudin, Function analysis, *McGraw-Hill*, (1973).
- [34] X. Ros-Oton and J. Serra, The extremal solution for the fractional Laplacian, *Calc. Var.* 50, 723-750 (2014).
- [35] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl.* 101(3), 275-302 (2014).
- [36] L. Véron, Singular solutions of some nonlinear elliptic equations, *Nonlinear Analysis: Theory, Methods & Applications* 5(3), 225-24 (1981).
- [37] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, Vol. I, 593-712, *Handb. Differ. Equ. North-Holland, Amsterdam* (2004).